On the Shape Factor for a Hollow, Square Cylinder

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The calculation of heat flow through the walls of long hollow cylinders of arbitrary cross section is of practical interest in the design of chemical process equipment. Simple analytical methods for computing the shape factors in such problems do not exist, but approximate shape factors have been obtained by electrical and finite-difference analog and, more recently, by semi-analytical series techniques. The purpose of this paper is to use a boundary residual technique to supplement the existing methods and to bound the error in the shape factor.

METHOD

The temperature field in the cylinder of cross section shown in Figure 1 may be written in the form

$$T(\rho, \theta) = A_0 + A_1 \ln \rho + \sum_{n=1}^{N} (a_n \rho^{4n} + b_n \rho^{-4n}) \cos 4n\theta \tag{1}$$

where the normalization

$$t(\rho, \theta) = t_i + (t_o - t_i)\, T(\rho, \theta) \tag{2}$$

has been used. The value of N is taken to be finite and the solution satisfying the inner boundary condition is

$$T(\rho, \theta) = a_0 \ln(\rho/\rho_i) + \sum_{n=1}^{N} a_n(\rho^{4n} - \rho_i^{8n} \rho^{-4n}) \cos 4n\theta \tag{3}$$

The remaining constants are obtained from the outer boundary condition so that

$$1 = a_0 \ln(\rho_o(\theta)/\rho_i) + \sum_{n=1}^{N} a_n(\rho_o(\theta)^{4n} - \rho_i^{8n} \rho_o(\theta)^{-4n}) \cos 4n\theta . \tag{4}$$

The methods of Schneider (1955) or Sparrow and Haji-Sheikh (1968) are candidates for the approximate calculation of the coefficients that satisfy this equation, but here we prefer to use a simple Fourier expansion. It is known [compare Petrovsky (1954), p. 141] that two functions are essentially identical if their Fourier coefficients are the same. Using this, the function on the right in Equation (4) will be approximately equal to that on the left if we set the first $N + 1$ Fourier coefficients of each side equal. Using the set of functions,

$$\psi_m(\theta) = \cos 4m\theta; \quad m = 0, 1, \ldots, N, \tag{5}$$

the solution of the set of algebraic equations

$$\int_0^{\pi/4} \psi_m(\theta)\, d\theta = a_0 \int_0^{\pi/4} \ln(\rho_o(\theta)/\rho_i)\, \psi_m(\theta)\, d\theta + \sum_{n=1}^{N} a_n \int_0^{\pi/4} [\rho_o(\theta)^{4n} - \rho_i^{8n} \rho_o(\theta)^{-4n}] \cos 4n\theta\, \psi_m(\theta)\, d\theta, \quad m = 0, \ldots, N, \tag{6}$$

where

$$\rho_o(\theta) = (\cos\theta)^{-1} \quad 0 \leqq \theta \leqq \pi/4$$

accomplishes this task.

This has been done for $R_i/R_o \leqq .975$ and for N up to 16. The resulting coefficients converge very rapidly for $R_i/R_o \lesssim 0.8$ and they converge slower but not unreasonably for $R_i/R_o \lesssim .975$. The total heat flux per unit length of cylinder is obtained by an integral over the inner boundary yielding

$$Q = 2\pi a_0\, k\, (t_i - t_o)$$

so the shape factor S defined by

$$S = 2\pi a_0$$

is seen to depend only upon the value of a_0.

For a one-term solution, the one remaining Equation (6) can be solved for the coefficient a_0 in closed form with the aid of Gradshteyn and Ryzhik (1965). For this case, the shape factor is

$$S = 2\pi\, \{-\ln(R_i/R_o) + .110\}^{-1}. \tag{7}$$

To obtain a more accurate value of S, the number of retained terms N is increased, the integrals in (6) are evaluated numerically by Simpson's rule, and the equations are solved by the Gauss-Jordan method.

Tests of the accuracy of the solution are of interest and, in the present case, two tests were used. First, the errors in the approximate solution for the temperature field can be bounded. The maximum modulus theorem [compare Petrovsky (1954), p. 169] guarantees that the maximum and minimum values of the exact solution for the temperature occur on the boundary. Since the approximate solu-

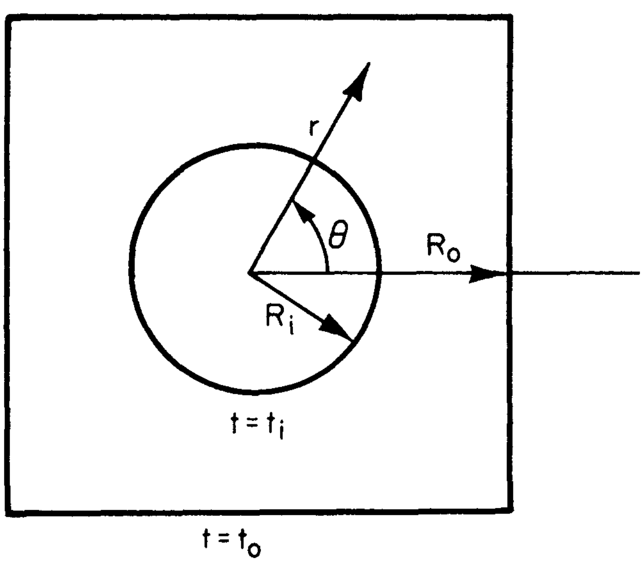

Fig. 1. The coordinate system and boundary conditions.

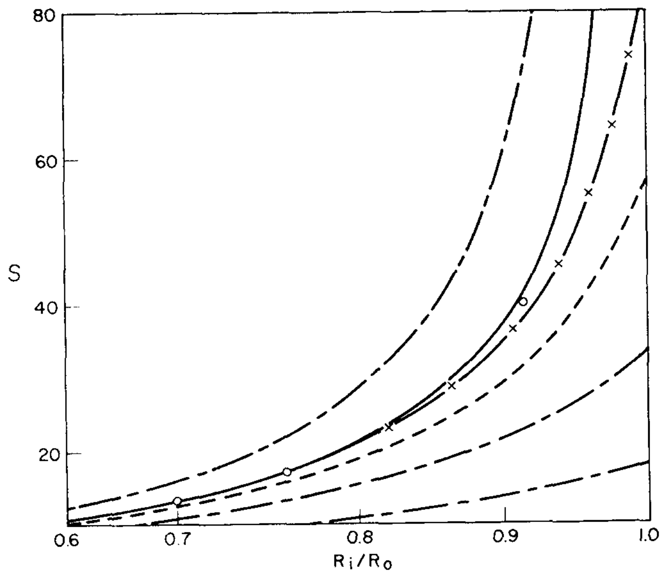

Fig. 2. Comparison of shape factors as functions of radius ratio. ———— *present numerical solution;* — — — — — present one-term solution; ——+—— Balcerzak and Raynor (1961); ○ Smith et al (1958); and ——— — ——— geometrical approximations.

tion satisfies the differential equation exactly, this implies that the maximum difference between the approximate and the exact solutions also occurs on the boundary. Once the series coefficients have been computed, a numerical search for the maximum error in satisfying the boundary condition bounds the error in the temperature field. In each case computed here, the error is less than 0.1%.

The second check is on the accuracy of the shape factor. For each case of R_i/R_o computed, N was increased until the value of a_N was less than 10^{-5}. For these larger values of N, the values of a_0 were observed to change in only the fourth decimal place. This (heuristically) implies that the resulting value for the shape factor is accurate to within 0.1% also.

RESULTS

The results for the shape factor are shown in Figure 2 along with previous results. Shih (1970a) considered this problem by the method of point-matching with the origin of the coordinate system taken in the corner of the square and, in that case, the boundary condition on the circular boundary was not satisfied exactly. Balcerzak and Raynor (1961) utilized a conformal transformation technique that amounts to point-matching with the origin taken as in the present analysis but they omitted terms in ρ^{-4n}. The experimental points of Smith et al. (1958) are shown by circles. The present numerical results, indicated by a solid line, are almost identical to those of Balcerzak and Raynor (1961) for $R_i/R_o \leq 0.7$ and to those of Shih (1970a) with the errata (Shih (1970b)) for $R_i/R_o < 0.6$. The one-term closed form solution (7), shown as a dashed line, gives acceptable results if R_i/R_o is not too large and it is easy to use. This accuracy is due to the fact that this solution satisfies the boundary condition on the outside surface on the average. Errors in evaluating the integral for the shape factor, therefore, largely cancel out.

Since the shape factor essentially is a measure of the cross sectional area conducting the heat divided by a path length, it can be estimated by geometrical arguments. Upper and lower bounds on the shape factor can be identified with circular cylinders whose outer boundaries are inscribed and circumscribed in the square so that an upper bound on S is

$$S = 2\pi \{-\ln(R_i/R_o)\}^{-1}$$

and a lower bound is

$$S = 2\pi \{-\ln(R_i/R_o) + .5 \ln 2\}^{-1}.$$

A closer estimate may be obtained by using the shape factor of a circular cylinder whose outer radius is the mean of the circumscribed and inscribed circles so that

$$S = 2\pi \{-\ln(R_i/R_o) + \ln(1 + 2^{1/2}) - \ln 2\}^{-1}.$$

These estimates are useful since they are easy to compute. They are also plotted in Figure 2 and their relative accuracies can be obtained therefrom.

All of the approximate formulae for the shape factor merge with the numerical results obtained here for the smaller values of the radius ratio R_i/R_o. However, for the larger values of R_i/R_o, the error bounds derived from the numerical solution lie within the approximate formulae.

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Professors Bryant and Pressman at Louisiana State University kindly pointed out the usefulness of the geometrical estimates for the shape factor.

NOTATION

A_0, A_1, a_n, b_n = constants in series solution for temperature
k = thermal conductivity of material between two boundaries
m, n = integers, ordering parameters
N = number of retained terms in series solution
Q = heat flux per unit length of cylinder
R_i = radius of inner boundary
R_o = shortest radial distance to outer boundary
$R_o(\theta)$ = radial distance to outer boundary
S = shape factor
$T(\rho, \theta)$ = nondimensional temperature defined by Equation (2)
t_i = temperature on inner boundary
t_o = temperature on outer boundary
ρ = r/R_o, nondimensional radial coordinate
ρ_i = nondimensional radius of inner boundary
$\rho_o(\theta)$ = nondimensional radial distance to outer boundary
$\psi_m(\theta)$ = orthogonal function set
θ = angular coordinate

LITERATURE CITED

Balcerzak, M. J., and S. Raynor, "Steady state temperature distribution and heat flow in prismatic bars with isothermal boundary conditions," *Intern. J. Heat Mass Transfer*, **3**, 113 (1961).

Gradshteyn, I. S., and I. M. Ryzhik, Tables of Integrals, Series, and Products, p. 535, Academic Press, New York (1965).

Petrovsky, I. G., Lectures on Partial Differential Equations, Interscience, New York (1954).

Shih, F. S., "On the temperature of a square column embedding a heating cylinder," *AIChE J.*, **16**, 134 (1970).

———., errata, *ibid.*, 1109.

Schneider, P. J., Conduction Heat Transfer, pp. 207-212, Addison-Wesley, Reading, Mass. (1955).

Smith, J. C., J. E. Lind, Jr., and D. S. Lermond, "Shape factors for conductive heat flow," *AIChE J.*, **4**, 330 (1958).

Sparrow, E. M., and A. Haji-Sheikh, "Transient and steady heat conduction in arbitrary bodies with arbitrary boundary and initial conditions," *J. Heat Transfer, Trans. ASME*, **C90**, 103 (1968).

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